ASSIGNMENT SET - I

Department of Mathematics

Mugberia Gangadhar Mahavidyalaya

B.Sc Hon.(CBCS)

Mathematics: Semester-VI

Paper Code: C14T

[Ring Theory and Linear Algebra - 2]

1. Answer all questions

(a) Consider the integral domain $\mathbb{Z}[\sqrt{-3}]$ then show that

(A) 1 and -1 are the only units in this integral domain.

(B) $1+\sqrt{-3}$, 2 are irreducible element in this integral domain.

(C) But none of $1+\sqrt{-3}$s and 2 is prime there.

(b) If I is an ideal of a ring *R*, prove that *I[x]* is an ideal of the polynomial ring *R[x]*.

(c) If $f(x)$ is a polynomial in $F(x)$ of degree 2 or 3 , then show that $f(x)$ is reducible over the field F iff it has a zero in F.

(d) Let *R* be an integral domain. Prove that *R* and *R[x]* have the same characteristic.

(e) Let R be a commutative ring with unity 1. Describe, $<x>$ the ideal of *R[x]* generated by *x.*

(f) Let $f(x)=x^4+3x^3+2x^2+2$ and $g(x)=x^2+2x+1$ $\in$**Q**[x]. Find the unique polynomials q(x), r(x) $\in$**Q**[x] such that *f(x)=q(x)g(x)+r(x)* , where either *r(x) = 0 or 0*$\leq \deg r(x) < \deg g(x)$.

(g) Let $f(x)=x^5+x^4+x^3+x+[3]$, $g(x)=x^4+x^3+[2]x^2+[2]x \in Z_5[x]$. Find q(x), r(x) $\in Z_5[x]$ such that $f(x)=q(x)g(x)+r(x)$, where either *r(x) = 0 or 0*$\leq \deg r(x) < \deg g(x)$.

(h) Let R = **Z⊕Z.** Show that the polynomial *(1,0)x* in *R[x]* has infinitely many roots in *R*.

(i) Show that the polynomial ring $Z_4[x]$ over the ring Z_4 is infinite, but $Z_4[x]$ is of finite characteristic.

(j) In the ring $Z_8[x]$, show that *[1] + [2]x* is a unit.

(k) Let R be a commutative ring with 1 and $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x]$. If *f(x)* is a unit in R[*x*], prove that a_0 is a unit in R and a_i is nilpotent for all $i = 1, 2, \ldots, n$.

(l) Find all units of **Z**[x].

(m) Find all units of $\mathbf{Z}_6$[x].

(n) Let R be an integral domain. Prove that the units of R[x] are contained in R.

2. Answer all question.

a) Let $f(x) = x^3 + 2x^2 + 1$ and $g(x) = 2x^2 + x + 2$. Check whether *f(x)* and *g(x)* reducible/irreducible .

b) In which the following fields, the polynomial $x^3 - 312312x + 123123$ is irreducible in $[x]$?

(A) The field F_3 with 3 elements.

(B) The field Q of rational numbers

(C) The field F_{13} with 13 elements.

c) Which of the following is an irreducible factor of $x^{12} - 1$ over Q ?

(A) $x^8 + x^4 + 1$

(B) $x^4 + 1$

(C) $x^4 - x^2 + 1$

d) Let R be the ring $Z[x]/((x^2 + x + 1)(x^3 + x+1))$ and I be the ideal generated by 2 in R. Calculate the cardinality of the ring R.

3. Answer all questions.

a) Define the Euclidean domain. If *F* is field, then the polynomial ring *F[x]* is a Euclidean domain.

b) Define principal ideal domain (PID). Prove that every Euclidean domain is a principal ideal domain (PID).

c) Let R be a commutative ring with 1. Then the following conditions are equivalent.

(A) R is a field.

(B) R[x] is a Euclidean domain.
(C) R[x] is a PID.

d) Let R be the Euclidean domain such that R is not a field. Then show that the polynomial ring R[x] is always a unique factorization domain (UFD) , but not a principal ideal domain (PID).
e) Check whether the ring $\mathbf{Q}[X,Y]/< X >$ is PID or not.
f) Show that $Z[\sqrt{n}]$ is a Euclidean domain for n= -1, -2, 2, 3.

4. Answer all questions.

a) Define the dual space of a vector space V. If V is a vector space of dimension n over a field F, then the dimension of its dual space is also n.
b) Let V be a finite dimensional vector space with dual space V^*, then every ordered basis for V^* is the dual basis for some basis for V.
c) Define the minimal polynomial, annihilator.
d) State and prove **Cayley Hamilton** theorem.
e) If A and B are similar matrices with entries from K, then prove that $m_A(x) = m_B(x)$.
f) Let *V* be vector space over *K*, let *p(x)* ∈ *K[x]* and let T ∈ L(V). Show that λ is an eigenvalue of T iff $p(\lambda)$ is an eigenvalue of $p(T)$.
g) Let V be vector space over R of dimension n=3 and let T ∈ L(V). Show that V has a non zero proper T-invarient subspace.

5. Answer all questions

a) Define inner product space. Let V be vector space over F, show that the sum of two inner products on V is an inner product on V.
b) Is the difference of two inner products an inner product?
c) Show that a positive multiple of an inner product is an inner product.
d) Verify that the standard inner product on F^n is an inner product.
e) Define orthogonal and orthogonal complement in an inner product space.
f) Find the orthogonal complement of the subspace P , generated by the vectors (1,1,0) and (0,1,1) in $\mathbb{R}^3$.
g) **Theorem:** An orthogonal set of non zero vectors is linearly independent.
h) **Theorem:** Let V be an inner product space and let $\beta\_1, \beta\_2, \ldots, \beta\_n$ be any independent vectors in V. Then one may construct orthogonal vectors $\alpha_1, \alpha_2, \ldots \alpha_n$ in V such that for each k = 1,2, …, n the set $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ is a basis for the subspace spanned by $\beta_1, \beta_2, \ldots, \beta_k$.

i) Prove that every finite dimensional inner product space has an orthonormal basis.

j) State and prove **Gram-Schmidt orthonormalization.**

k) Let W be a subspace of finite dimensional inner product space V. Then $W^{\perp\perp} = W$.

l) **Theorem:** Let W be a finite dimensional subspace of an inner product space V and let E be the orthogonal projection of V on W. Then E is an idempotent linear transformation of V onto W, $W^{\perp}$ is the null space of E, and $V = W \oplus W^{\perp}$.

m) **Theorem:** Let V be finite dimensional inner product space, and f be a linear functional on V. Then there exist a unique vector β in V such that $f(\alpha) = (\alpha|\beta)$ for all α in V.

n) Define Self adjoint and normal operators.

o) **Proposition:** Let T be self adjoint operator on an inner product space V.

(i) For all $v \in V$, (Tv, v) is a real number.

(ii) If $(Tv, v) = 0$, for all $v \in V$, then $T \equiv 0$.

(iii) If $T^k v = 0$ for some k > 1, then $Tv = 0$.

Further if V is finite dimensional, then

(iv) all roots of characteristic polynomial of T are real;

(v) eigenvectors corresponding to distinct eigenvalues are orthogonal.

p) **Proposition:** Let T be a normal operator on an inner product space V over F.

(i) $\|Tv\| = \|T^*v\|$ for all $v \in V$.

(ii) If for $v \in V$, $Tv = \lambda v$, $\lambda \in F$, then $T^*v = \bar{\lambda} v$.

(iii) Eigenvectors corresponding to distinct eigenvalues of T are orthogonal.

(iv) If $T^k v = 0$, for some k ≥ 1, then $Tv = 0$.

q) Prove the **Spectral Theorem.** Let T be a triangulable linear operator on a finite dimensional inner product space V over F. Then T is normal iff V has an orthonormal basis consisting of eigenvectors of T.

r) Let V be finite dimensional complex inner product space and let $T \in L(V)$. Then T is normal if and only if T is orthogonally diagonalizable.

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